CONVECTIVE HEAT TRANSFER FOR A NONLINEAR VISCOPLASTIC MEDIUM IN A CIRCULAR PIPE

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Heat transfer is considered for a viscoplastic material of nonlinear type flowing with dissipation in a circular tube with developed laminar flow and boundary conditions of the first kind.

There is much interest in heat transfer in the laminar flow of systems in circular tubes. The problem has been considered in reasonable detail for Newtonian liquids [1]. See [2] for the flow of a non-Newtonian liquid subject to a power law and dissipative heating. Forced laminar convection of heat in a viscoplastic medium of Shvedov–Bingham type without dissipation was examined in [3], and the case of internal viscous heating was studied in [4-7]. However, all these papers give only a restricted range of eigenvalues, eigenfunctions, and basic coefficients for the Fourier expansion. Calculations for viscoplastic media have been given for three or four values of the plasticity parameter, which makes it difficult to use them for other conditions, particularly for the flow near the inlet to a tube.

Here we solve the Gretz – Nusselt problem with allowance for dissipation for a viscoplastic composition of nonlinear type described by the rheological equation

$$\tau^{\frac{1}{n}} = \tau_0^{\frac{1}{n}} + (\eta_p\dot{\gamma})^{\frac{1}{m}}, \tag{1}$$

which was proposed in [8].

In (1), τ and τ_0 are the shear stress and limiting shear stress, η_p is the analog of the plastic viscosity, $\dot{\gamma}$ is the shear rate, and n and m are real numbers that can take any values.

Here we use the curves of (1) with the condition n = m, i.e.,

$$\tau^{\frac{1}{n}} = \tau_0^{\frac{1}{n}} + \left(\eta_p\dot{\gamma}\right)^{\frac{1}{n}}. \tag{1a}$$

Direct integration of (1a) with the condition of adhesion at the wall [9] gives the velocity distribution across the tube with a steady laminar flow and constant physical properties for the liquid:

$$u(\rho) = \frac{aR^2}{2\eta_p}\sum_0^n (-1)^k C_n^k \frac{n}{2n-k}\sigma_0^{k/n}\left(1-\rho^{\frac{2n-k}{n}}\right). \tag{2}$$

Here $\rho = r/R$, r is the current radius, R is the radius of the tube, $a = \Delta P/l$ is the pressure fall along a length l, $\sigma_0 = \tau_0/\tau_W = r_0/R$ is the dimensionless radius of the quasirod zone, τ_W is the shear stress at the wall, C_n^k are binomial coefficients, and n is as appears in (1a).

The shear stress is less than τ_0 near the axis, and the material moves along the tube as a solid at the constant velocity

$$u(\sigma_0) = u_{\max} = \frac{aR^2}{2\eta_p}\sum_0^n (-1)^k \frac{nC_n^k\sigma_0^{k/n}}{2n-k}\left(1-\sigma_0^{\frac{2n-k}{n}}\right). \tag{3}$$

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The velocity function is put in dimensionless form as

$$\frac{u(r)}{u_{\max}}=\varphi=1-\frac{\sum\limits_{0}^{n}(-1)^k\frac{nC_n^k}{2n-k}\,\sigma_0^{k/n}\left(\rho^{\frac{2n-k}{n}}-\sigma_0^{\frac{2n-k}{n}}\right)}{\tilde{u}_{\max}},\tag{4}$$

where

$$\tilde{u}_{\max}=\sum_{0}^{n}(-1)^k\frac{nC_n^k\sigma_0^{\frac{k}{n}}}{2n-k}\left(1-\sigma_0^{\frac{2n-k}{n}}\right).$$

Equations (2)-(4) are reasonably convenient for computation if n does not exceed 4; larger values lead to reduced accuracy because the series are sign-varying and the terms in the summation are similar in magnitude. The error is particularly large for $\sigma_0\to 1$. Formula (4) has [9] been put in the form

$$\varphi=1-\frac{\left(\rho^{\frac{1}{n}}-\sigma_0^{\frac{1}{n}}\right)^{n+1}\left[\sigma_0^{\frac{n-1}{n}}+C_{2n,1}^n\rho^{\frac{n-1}{n}}+\sum\limits_{2}^{n-1}C_{n+k-1}^n(\sigma_0\rho)^{\frac{n-k}{n}}\right]}{\left(1-\sigma_0^{\frac{1}{n}}\right)^{n+1}\sum\limits_{0}^{n}C_{n+k-1}^n\sigma_0^{\frac{n-k}{n}}},\tag{5}$$

which enables one to do the calculation of φ for any n and σ_0 with the required degree of accuracy.

The following is the energy equation for one-dimensional steady-state flow with constant physical characteristics incorporating viscous dissipation but neglecting the axial leakage of heat by conduction:

$$\rho^* C_p u(r)\frac{\partial t}{\partial z}=\frac{\lambda}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right)+\tau_{rz}\left(-\frac{du}{d\rho}\right),\tag{6}$$

where ρ^*, C_p, λ are respectively the density, specific heat, and thermal conductivity of the medium, while z is the axial coordinate.

This solution was compared with that in [8] by assuming that the liquid enters the heat-transfer section at a temperature t_0 constant over the cross section and with the velocity profile described by (2). Then we have the following system of equations and boundary conditions if we neglect the mass forces for the problem of (1a) for a circular tube with a constant wall temperature t_W:

$$\frac{\rho^* C_p u_{\max}R}{\lambda}\varphi\frac{\partial t}{R\,\partial z}=\frac{1}{R^2\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\frac{\partial t}{\partial\rho}\right)+\tau_{rz}\left(-\frac{du}{d\rho}\right)\frac{1}{R\lambda},$$
$$t(0,\ \rho)=t_0;\ t(z,\ 1)=t_w;\ \left.\frac{\partial t}{\partial\rho}\right|_{\rho=0}=0,\tag{7}$$

where

$$\varphi=\frac{u(r)}{u_{\max}}.$$

We substitute in (7) for $u_{\max}$ from (3) and introduce

$$w=\frac{aR^2}{8\eta_p}\ \text{and}\ \mathrm{Pe}=\frac{2w\rho^* C_p R}{\lambda}.$$

Then the factor $\rho^* C_p u_{\max}R/\lambda$ becomes

$$\frac{\rho^* C_p u_{\max}R}{\lambda}=\mathrm{Pe}\frac{\left(1-\sigma_0^{\frac{1}{n}}\right)^{n+1}}{C_{2n-1}^n}\sum_{1}^{n}C_{n+k-1}^n\sigma_0^{\frac{n-k}{n}}=\mathrm{Pe}\,F(n,\ \sigma_0),$$

where, from [9],

$$F(n,\ \sigma_0)=\frac{\left(1-\sigma_0^{\frac{1}{n}}\right)^{n+1}}{C_{2n-1}^n}\sum_{1}^{n}C_{n+k-1}^n\sigma_0^{\frac{n-k}{n}}=\frac{\left(1-\sigma_0^{\frac{1}{n}}\right)^{n+1}}{C_{2n-1}^n}\sum_{0}^{n-1}C_{2n-k-1}^n\sigma_0^{\frac{k}{n}}.\tag{8}$$

We use the relations

$$\tau_{\rho z}=\frac{aR}{2}\rho;\ -\frac{du}{d\rho}=\frac{aR^2}{2\eta_p}\left(\rho^{\frac{1}{n}}-\sigma_0^{\frac{1}{n}}\right)^n;\ D=\frac{16\eta_p\omega^2}{\lambda(t_0-t_w)}$$

and the dimensionless variables

$$X=\frac{z}{2R\,\mathrm{Pe}}\ \text{and}\ \vartheta=\frac{t-t_w}{t_0-t_w},$$

to transform (7) finally to

$$\frac{1}{2}F(n,\ \sigma_0)\varphi(\rho)\frac{\partial\vartheta}{\partial X}=\frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\vartheta}{\partial\rho}\right)+D\rho\left(\rho^{\frac{1}{n}}-\sigma_0^{\frac{1}{n}}\right)^n,$$

$$\vartheta(X,\ 1)=\left.\frac{\partial\vartheta}{\partial\rho}\right|_{\rho=0}=0;\ \vartheta(0,\ \rho)=1. \tag{9}$$

We put the solution to (9) as the sum of two functions: $\vartheta=\vartheta_1+\vartheta_2$, of which the first $\vartheta_1(X,\ \rho)$ satisfies the corresponding homogeneous equation with the boundary conditions of (9), while the second satisfies the inhomogeneous equation with zero boundary conditions.

It is clear that ϑ_1 in this case coincides with the Gretz–Nusselt solution [1]:

$$\vartheta_1(X,\ \rho)=\sum_0^\infty C_k\exp\left(-\frac{2\beta_k^2}{F}X\right)\psi_k(\beta_k\rho), \tag{10}$$

where β_k and ψ_k are the eigenvalues and the functions for the Sturm–Liouville problem

$$(\rho\psi_k')'+\beta_k^2\rho\varphi(\rho)\psi_k=0,$$
$$\psi_k(1)=\psi_k'(0)=0, \tag{11}$$

where the C_k are the coefficients in the expansion of $\vartheta_0(0,\ \rho)=1$ with respect to the eigenfunctions ψ_k with the weight $\rho\varphi(\rho)$.

The particular solution ϑ_2 to (9) we put as a series in the eigenfunctions ψ_k with the coefficients dependent on the argument X:

$$\vartheta_2(X,\ \rho)=\sum_0^\infty A_k(X)\psi_k(\beta_k,\ \rho). \tag{12}$$

We substitute for $\vartheta_2(X,\ \rho)$ in (9) to get

$$\frac{F}{2}\varphi(\rho)\sum_0^\infty\frac{dA_k}{dX}\psi_k=\frac{1}{\rho}\sum_0^\infty A_k(\rho\psi_k')'+D\rho\left(\rho^{\frac{1}{n}}-\sigma_0^{\frac{1}{n}}\right)^n \tag{13}$$

or, using (11),

$$\frac{F}{2}\sum_0^\infty\frac{dA_k}{dX}\psi_k+\sum_0^\infty A_k\beta_k^2\psi_k=\frac{D\rho\left(\rho^{\frac{1}{n}}-\sigma_0^{\frac{1}{n}}\right)^n}{\varphi(\rho)}. \tag{13a}$$

We expand $\rho(\rho^{1/n}-\sigma_0^{1/n})/\varphi(\rho)$ as a Fourier series with respect to the ψ_k to get

$$\frac{F}{2}\sum_0^\infty\frac{dA_k}{dX}\psi_k+\sum_0^\infty A_k\beta_k^2\psi_k=D\sum_0^\infty a_k\psi_k, \tag{13b}$$

where

$$a_k=\frac{\int_0^1\rho^2\left(\rho^{\frac{1}{n}}-\sigma_0^{\frac{1}{n}}\right)^n\psi_k\,d\rho}{\int_0^1\rho\varphi(\rho)\psi_k^2\,d\rho}. \tag{14}$$

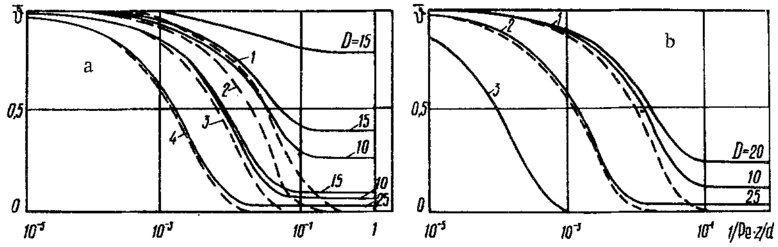

Fig. 1. Variation of the mean-mass temperature along the length of the tube: a) n = 1 (Shvedov–Bingham model): 1) $\sigma_0 = 0$ (Newtonian liquid); 2) 0.3; 3) 0.6; 4) 0.8; b) $\sigma_0 = 0.5$; 1) n = 1; 2) 2; 3) 3 (D = 0; 25).

As (13b) should be obeyed for any ψ_k, the coefficients to the terms containing the ψ_k are zero, i.e.,

$$\frac{F}{2}\cdot\frac{dA_k}{dX} + A_k\beta_k^2 = Da_k;\ A_k(0) = 0. \tag{15}$$

The solution to (15) is the function

$$A_k(X) = D\frac{a_k}{\beta_k^2}\left[1 - \exp\left(-\frac{2\beta_k^2}{F}X\right)\right]. \tag{16}$$

Then (12) is put as

$$\vartheta_2(X,\ \rho) = D\sum_0^\infty \frac{a_k}{\beta_k^2}\left[1 - \exp\left(-\frac{2\beta_k^2}{F}X\right)\right]\psi_k(\beta_k,\ \rho). \tag{17}$$

Formulas (10) and (17) give the general solution to the problem of (9):

$$\vartheta = \vartheta_1 + \vartheta_2 = \sum_0^\infty C_k \exp\left(-\frac{2\beta_k^2}{F}X\right)\psi_k + D\sum_0^\infty \frac{a_k}{\beta_k^2}\left[1 - \exp\left(-\frac{2\beta_k^2}{F}X\right)\right]\psi_k. \tag{18}$$

From (18) we isolate the quantity

$$\vartheta_* = D\sum_0^\infty \frac{a_k}{\beta_k^2}\psi_k(\beta_k,\ \rho) \tag{19}$$

or

$$\frac{t - t_w}{16\eta_p w^2/\lambda} = \sum_0^\infty \frac{a_k}{\beta_k^2}\psi_k, \tag{19a}$$

which is the temperature rise due to the frictional heat [10].

The first term in (18) falls from 1 for X = 0 to zero for $X \to \infty$, whereas the second series increases with X from zero for X = 0 to the maximum value of ϑ_* for $X \to \infty$; if there is no dissipative heat production, i.e., if D = 0, then (18) reduces to (10).

The temperature distribution is known, so it is easy to determine the mean-mass temperature of the liquid in a given cross section [1]:

$$\bar{\vartheta} = \frac{\bar{t} - t_w}{t_0 - t_w} = \frac{2u_{\max}}{\bar{u}}\int_0^1 \vartheta\varphi(\rho)\rho\, d\rho. \tag{20}$$

In turn

$$\bar{u} = 2\int_0^1 \rho\, u(\rho)\, d\rho,$$

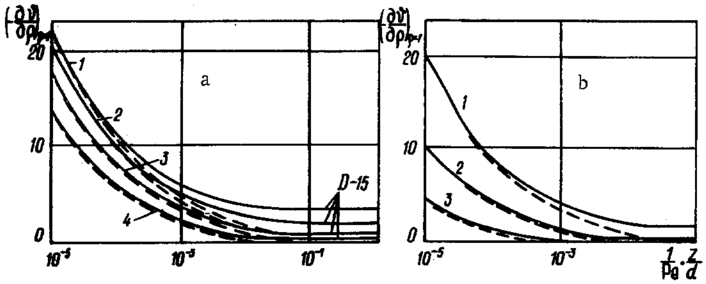

Fig. 2. Variation of the temperature gradient at the wall along the length of the tube: a) n = 1: 1) $\sigma_0 = 0$; 2) 0.3; 3) 0.6; 4) 0.8; b) $\sigma_0 = 0.5$ (D = 0; 25): 1) n = 1; 2) 2; 3) 3.

whence

$$\frac{\bar{u}}{2\bar{u}_{\max}} = \int_0^1 \rho\varphi(\rho)\,d\rho. \tag{21}$$

Substitution of (21) into (20) gives

$$\bar{\vartheta} = \frac{\int_0^1 \rho\,\vartheta\,\varphi(\rho)\,d\rho}{\int_0^1 \rho\varphi(\rho)\,d\rho}. \tag{22}$$

We substitute from [9] the values for $\bar{u}$ and $u_{\max}$ to represent the denominator of (22) as

$$I_1 = \int_0^1 \rho\varphi(\rho)\,d\rho = \frac{C_{2n-1}^{n}\sum_{0}^{3n-1} C_{4n-k-1}^{n}\,\sigma_0^{\frac{k}{n}}}{4C_{4n-1}^{n}\sum_{0}^{n-1} C_{2n-k-1}^{n}\,\sigma_0^{\frac{k}{n}}}. \tag{23}$$

We substitute into (22) the value of ϑ from (18) and use (11) to get

$$\bar{\vartheta} = -\frac{1}{I_1}\sum_{0}^{\infty}\frac{1}{\beta_k^2}\left\{C_k\exp\left(-\frac{2\beta_k^2}{F}X\right) + D\frac{a_k}{\beta_k^2}\left[1-\exp\left(-\frac{2\beta_k^2}{F}X\right)\right]\right\}\left(\frac{d\psi_k}{d\rho}\right)_{\rho=1}$$

or

$$\bar{\vartheta} = \frac{2}{I_1}\sum_{0}^{\infty}\frac{1}{\beta_k^2}\left\{B_k\exp\left(-\frac{2\beta_k^2}{F}X\right) + D\frac{b_k}{\beta_k^2}\left[1-\exp\left(-\frac{2\beta_k^2}{F}X\right)\right]\right\}, \tag{24}$$

where

$$B_k = -\frac{1}{2}C_k\left(\frac{d\psi_k}{d\rho}\right)_{\rho=1};\quad b_k = -\frac{1}{2}a_k\left(\frac{d\psi_k}{d\rho}\right)_{\rho=1} \tag{25}$$

Equation (24) defines the mean-mass temperature of the medium as a function of the reduced length $X = (1/Pe)(z/d)$, the dissipation parameter, and F. Figure 1 shows a family of curves for $\bar{\vartheta}$ as a function of X for n = 1 for a series of values of σ_0 and D. The curves for $\bar{\vartheta}(X, D, \sigma_0, n)$ for the various n and D with $\sigma = 0.5$ are shown in Fig. 2 (the broken lines correspond to the absence of dissipation, i.e., to the case of $D \cong 0$).

We ascribe the local heat-transfer coefficient to the temperature difference $t_w - \bar{t}$ to get for Nu the expression

$$\mathrm{Nu} = -\frac{2}{\bar{\vartheta}}\left(\frac{\partial\vartheta}{\partial\rho}\right)_{\rho=1}.$$

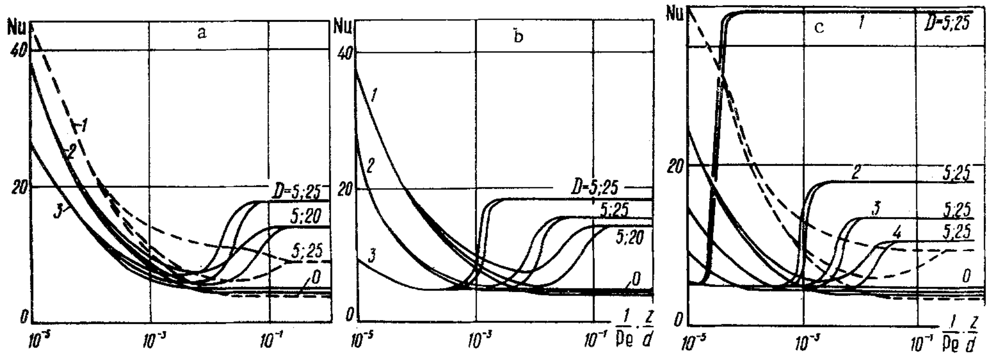

Fig. 3. Variation of local Nusselt number along the length of the tube: a) n = 1; 1) $\sigma_0 = 0$; 2) 0.5; 3) 0.08; b) $\sigma_0 = 0.5$: 1) n = 3; 2) 2; 3) 1; c) n = 3: 1) $\sigma_0 = 0.8$; 2) 0.5; 3) 0.3; 4) 0.1.

We substitute for ϑ from (18) and for $\bar{\vartheta}$ from (24) to get

$$\mathrm{Nu} = 2I_1 \frac{\sum_0^\infty \left(B_k - D\frac{b_k}{\beta_k^2}\right)\exp\left(-\frac{2\beta_k^2}{F}X\right) + D\sum_0^\infty \frac{b_k}{\beta_k^2}}{\sum_0^\infty \frac{1}{\beta_k^2}\left(B_k - D\frac{b_k}{\beta_k^2}\right)\exp\left(-\frac{2\beta_k^2}{F}X\right) + D\sum_0^\infty \frac{b_k}{\beta_k^4}}. \tag{26}$$

In Fig. 2a graphs of $-\partial\vartheta/\partial\rho|_{\rho=1}$ are shown for a series of values of σ_0 and D for n = 1 (the Shvedov–Bingham model); the broken curves correspond to the absence of dissipation. In Fig. 2b curves are given for $-\partial\vartheta/\partial\rho|_{\rho=1} = f(X, D, \sigma_0, n)$ for various values of n and D and at $\sigma_0 = 0.5$.

We can give formula (26) a somewhat different form for convenience in analysis; we use (19) to rewrite the expression (18) as

$$\vartheta = \sum_0^\infty \left[C_k \exp\left(-\frac{2\beta_k^2}{F}X\right) - D\frac{a_k}{\beta_k^2}\left(-\frac{2\beta_k^2}{F}X\right)\right]\psi_k + \vartheta_*. \tag{18a}$$

Then we have

$$\mathrm{Nu} = 2I_1 \frac{\sum_0^\infty \left(B_k - D\frac{b_k}{\beta_k^2}\right)\exp\left(-\frac{2\beta_k^2}{F}X\right) - \left(\frac{\partial\vartheta_*}{\partial\rho}\right)_{\rho=1}}{2\sum_0^\infty \frac{1}{\beta_k^2}\left(B_k - D\frac{b_k}{\beta_k^2}\right)\exp\left(-\frac{2\beta_k^2}{F}X\right) + \int_0^1 \vartheta_*\varphi(\rho)\rho\, d\rho}. \tag{26a}$$

For $X \to \infty$ the series in the numerator of (26a) becomes small in comparison with the temperature gradient at the wall, while the series in the denominator becomes small by comparison with $\int_0^1 \vartheta_*\varphi(\rho)\rho d\rho$, and Nu tends to the constant value

$$\mathrm{Nu}_\infty = -2I_1 \frac{\left(\frac{\partial\vartheta}{\partial\rho}\right)_{\rho=1}}{\int_0^1 \vartheta_*\varphi(\rho)\rho\, d\rho}. \tag{27}$$

To determine the limiting value of Nu in analytical form we represent ϑ_* as

$$\vartheta_* = D\sum_0^n (-1)^k C_n^k \frac{n}{4n-k}\sigma_0^{\frac{k}{n}}\left(1 - \rho^{\frac{4n-k}{n}} + \sigma_0^{\frac{4n-k}{n}}\ln\rho\right), \tag{28}$$

where n is as in (1a), in particular n = 1 for the Shvedov–Bingham model and n = 2 for the Casson model. We see that (19) and (28) are equivalent to the formulas of [10].

For a Newtonian liquid ($\sigma_0 = 0$), we have from (4), (23), and (28) that

$$\vartheta_* = \frac{1}{4} D(1-\rho^4); \quad \varphi(\rho) = 1-\rho^2; \quad I_1 = \frac{1}{4}; \quad \left(\frac{\partial \vartheta_*}{\partial \rho}\right)_{\rho=1} = -D;$$

$$\int_0^1 \vartheta_* \varphi(\rho) \rho \, d\rho = \frac{5}{96} D.$$

Then (27) shows that $Nu_\infty = 48/5 = 9.6$, which corresponds to the value found previously [11] by a solution of this problem for a Newtonian liquid.

The curves for Nu(z/Pe d, D, σ_0, n) for a Shvedov–Bingham medium for various σ_0 and D are shown in Fig. 3a; the broken lines in Fig. 3a represent the Nu(z/Pe d, D) curves for a Newtonian liquid.

In Fig. 3b the relationship Nu(z/Pe d, D, σ_0, n) is shown for various n and D with $\sigma_0 = 0.5$; it is clear that Nu increases with X more or less rapidly as a function of D and tends to the limiting value Nu_∞, which, in this case (non-Newtonian liquid) is a function of σ_0 and n.

For $\sigma_0 = 0.5$ (Fig. 3b), Nu_∞ takes the following values: 14 for n = 1, 16 for n = 2, and 18 for n = 3; i.e., Nu_∞ increases with the nonlinearity parameter. It is evident from Fig. 3c that Nu_∞ also increases with σ_0. Calculations were performed from (26) for n = 1–3 and $\sigma_0 = 0.1$–0.8 with steps of 0.1; these showed that Nu_∞ in the presence of dissipation increases from 9.8 to 18 for n = 1, from 10.2 to 34.5 for n = 2, and from 10.8 to 39.4 for n = 3. In the absence of dissipation, this same range of variation in σ_0 causes Nu_∞ to increase from 3.8 to 4.8 for n = 1, from 4.0 to 5.0 for n = 2, and from 4.2 to 5.2 for n = 3. For n of 4 and 5 we perform the calculations only for $\sigma_0 = 0.5$; in this case, we get for dissipation that $Nu_\infty = 20.7$ for n = 4 and 23 for n = 5, while in the absence of dissipation $Nu_\infty = 4.8$ and 4.9 respectively.

Then a non-Newtonian system described by (1a) in a circular tube has higher Nu_∞ than does a Newtonian liquid under the same conditions; the same may be said even for the absence of dissipation, but in this case the effects of the plasticity and nonlinearity parameters are much less (the Nu_∞ for a Newtonian liquid is 3.66 in the absence of dissipation).

Thermal stabilization occurs over a length at the end of which Nu_∞ (for given σ_0 and n) differs from the final value by not more than 1%; this length is dependent on D and decreases as the latter increases (Fig. 3).

These results indicate that dissipation in the form of heat causes a considerable increase in N_∞ and, consequently, in the local heat-transfer coefficients α_∞; this increase in due basically to the radical change in the temperature profile, which is due to the marked increase in the temperature gradients near the wall, where there is particularly marked mechanical energy conversion. Figure 3 shows that this rise is less pronounced at small differences from the inlet, but it is decisive from the point of minimum on the curve for Nu(z) = f(1/Pe · z/d, D, σ_0, n), which moves towards the start of the tube as D increases.

The iteration method of [12] was used to determine the first eigenvalues and eigenfunctions; the essence of this for (11) is as follows.

We multiply both parts of (11) by ψ_k and integrate from 0 to 1 to get

$$\int_0^1 \psi_k \, d(\rho\psi_k') = -\beta_k^2 \int_0^1 \rho\varphi(\rho)\psi_k^2 \, d\rho,$$

whence integration by parts gives

$$\beta_k^2 = \varepsilon_k = \frac{\int_0^1 \rho\psi_k' \, d\rho - \psi_k(\beta_k, 1)\,\psi_k'(\beta_k, 1)}{\int_0^1 \rho\varphi(\rho)\,\psi_k^2 \, d\rho}. \tag{29}$$

We assume for the zeroth approximation that

$$\varepsilon_k^{(0)} = \left[\frac{\pi\left(k+\frac{2}{3}\right)}{\int\limits_0^1 \sqrt{\varphi(\rho)}\, d\rho} \right]^2, \tag{30}$$

which is derived from the asymptotic solution of (11) [8].

We substitute for $\varepsilon_k^{(0)}$ into

$$(\rho\psi_k')' + \varepsilon_k^{(0)}\rho\varphi(\rho)\,\psi_k = 0 \tag{31}$$

and solve the Cauchy problem for this equation by the Runge–Kutta method with the boundary conditions $\psi_k'(0) = 0$, $\psi_k(0) = 1$ to get approximate values for the eigenfunctions at the points ρ_i in the interval $(0.1-\psi_k(\sqrt{\varepsilon_k^{(0)}}, \rho_i))$, and also for $\psi_k(\sqrt{\varepsilon_k^{(0)}}, 1)$, $\psi_k'(\sqrt{\varepsilon_k^{(0)}}, 1)$ and the norm $\int\limits_0^1 \rho\varphi(\rho)\psi_k^2(\sqrt{\varepsilon_k^{(0)}}, \rho)d\rho$.

Further, from

$$\varepsilon_k^{(p)} = \varepsilon_k^{(p-1)} - \frac{\psi_k(\sqrt{\varepsilon_k^{(p-1)}}, 1)\,\psi_k'(\sqrt{\varepsilon_k^{(p-1)}}, 1)}{\int\limits_0^1 \rho\varphi(\rho)\,\psi_k^2(\sqrt{\varepsilon_k^{(p-1)}}, \rho)\,d\rho} \tag{32}$$

we get the subsequent approximations $\varepsilon_k^{(p)}$.

Equation (32) applies from p = 1; each successive $\varepsilon_k^{(p)}$ is substituted in (31) to calculate the eigenfunction, the derivative of this, and the norm. The process converges sufficiently rapidly in the third or fourth approximation. The method enables one to derive not less than 5 or 6 first values for β_k and $\psi_k(\beta_k, \rho_i)$; the asymptotic solution is sufficiently accurate for $k \geq 4.5$, so subsequent computations of the eigenvalues and eigenfunctions, and also any relative quantities, should be done via the asymptotic formulas given in [8].

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